3 Matrix powers

SL Type I

Method

1. Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

   Calculate $\boldsymbol{M}^n$ for $n = 2, 3, 4, 5, 10, 20, 50$.

   Describe in words any pattern you observe.

   Use this pattern to find a general expression for the matrix $\boldsymbol{M}^n$ in terms of n.

2. Consider the matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{S} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

$$\boldsymbol{P}^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix};\ \boldsymbol{S}^2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

   Calculate $\boldsymbol{P}^n$ and $\boldsymbol{S}^n$ for other values of n and describe any pattern(s) you observe.

3. Now consider matrices of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

   Steps 1 and 2 contain examples of these matrices for $k = 1, 2$ and 3.

   Consider other values of k, and describe any pattern(s) you observe.

   Generalize these results in terms of k and n.

4. Use technology to investigate what happens with further values of k and n. State the scope or limitations of k and n.

5. Explain why your results hold true in general.

5 Stopping distances

SL Type II

Description

When a driver stops her car, she must first think to apply the brakes. Then the brakes must actually stop the vehicle.

The table below lists average times for these processes at various speeds.

Speed (kmh^{-1})	Thinking distance (m)	Braking distance (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

In this task you will develop individual functions that model the relationships between speed and thinking distance, as well as speed and braking distance. You will also develop a model for the relationship between speed and overall stopping distance.

Method

1. Use a GDC or graphing software to create two data plots: speed versus thinking distance and speed versus braking distance. Describe your results.
2. Using your knowledge of functions, develop functions that model the behaviours noted in step 1. Explain your work.
3. The overall stopping distance is obtained from adding the thinking distance to the braking distance. Create a data table of speed and overall stopping distance. Graph this data and describe the results.
4. Develop a function that models the relationship between speed and overall stopping distance. How is this function related to the functions obtained in step 2?
5. Overall stopping distances for other speeds are given below. Discuss how your model fits this data, and what modifications might be necessary.

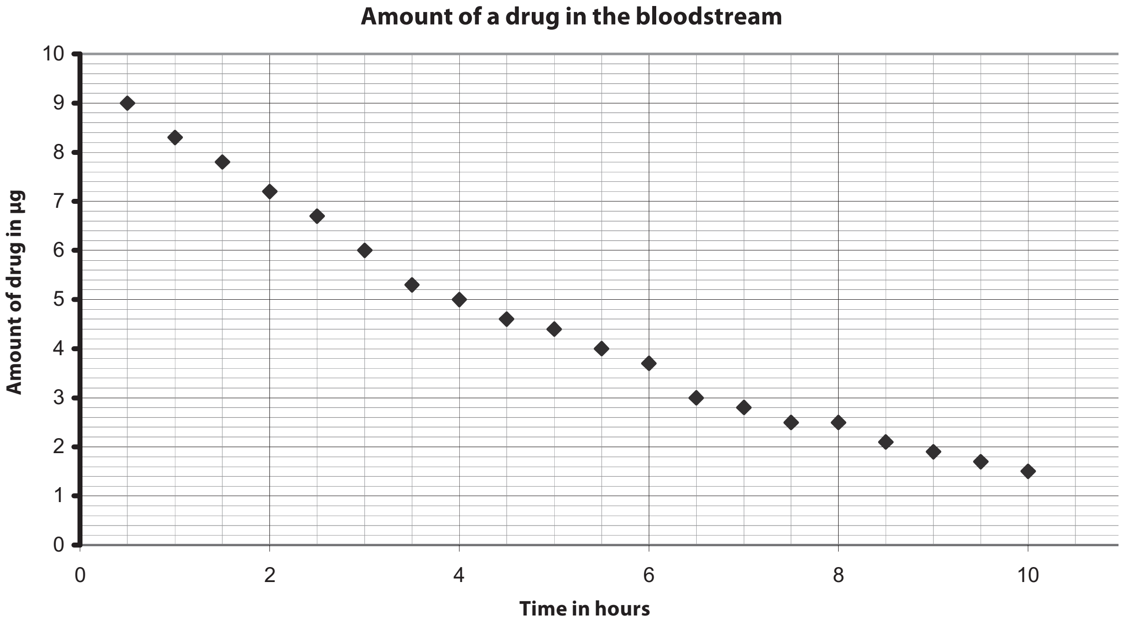
Speed (kmh^{-1})	Stopping distance (m)
10	2.5
40	17
90	65
160	180

8 Modelling the amount of a drug in the bloodstream

SL Type II

Description

The graph below records the amount of a drug for treating malaria in the bloodstream over the 10 hours following an initial dose of 10µg.



It seems that the rate of decrease of the drug is approximately proportional to the amount remaining.

Method

Part A

1. Use this information to help you find a suitable function to model this data.
2. Draw a graph of your function and compare your graph to the one above.
3. Comment on the suitability of the model.

Part B

A patient is instructed to take 10µg of this drug every six hours.

1. **Sketch** a diagram to show the amount of the drug in the bloodstream over a 24-hour period and state any assumptions made.
2. Use your GDC or graphing software and your model from part A to draw an accurate graph to represent this situation.
3. State the maximum and minimum amounts during this period.
4. Describe what would happen to these values over the next week if:
   (a) no further doses are taken
   (b) doses continue to be taken every six hours.

Introduction

This section contains examples of student work that have been produced from the tasks in the previous section.

Student	Title of task	Number of specimen portfolio task
A	Matrix powers	3
B	Matrix powers	3
C	Stopping distances	5
D	Modelling the amount of a drug in the bloodstream	8

The numbers of these tasks, listed in the third column, refer to the order in which they appear in the section entitled "Specimen portfolio tasks".

The assessment for each piece of work appears at the end of the work. The intention is to demonstrate the overall standards required for mathematics SL and to illustrate how the achievement levels for each criterion should be awarded.

The assessment was undertaken by a team of senior examiners and teachers. Each piece of work has been marked more than once and agreement reached on the achievement level to be awarded. Explanations are given for the award of each achievement level and, where appropriate, specific references to the work of the student have been made.

Teachers may wish to mark the student work themselves before reading the assessment to compare their standard of marking against that of the examiners.

Type I tasks

Matrix powers—student A

SL Type I

Matrix Powers

Lacks an introduction

1. $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$M^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+0 \\ 0+0 & 0+4 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Using GDC;

$M^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, $M^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

$M^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$, $M^{10} = \begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$

$M^{20} = \begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$

$M^{50} \doteq \begin{pmatrix} 1.13 \times 10^{15} & 0 \\ 0 & 1.13 \times 10^{15} \end{pmatrix}$ ✓

For each higher power of M the entries double.

So $M^2 = 2 \cdot M$

$M^3 = 2 \cdot M^2 = 2^2 \cdot M$

$M^4 = 2 \cdot M^3 = 2^3 \cdot M$

$\vdots$

$M^n = 2^{n-1} \cdot M$ ✓

— 1 —

2. $P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

just enough data

$P^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

by GDC $P^3 = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix} = 2 \begin{pmatrix} 18 & 14 \\ 14 & 18 \end{pmatrix} = 2^2 \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$

$P^4 = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix} = 2 \begin{pmatrix} 68 & 60 \\ 60 & 68 \end{pmatrix}$

$= 2^2 \begin{pmatrix} 34 & 30 \\ 30 & 34 \end{pmatrix} = 2^3 \begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix}$

I will continue taking powers of 2 out.

$P^5 = \begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix} = 2^4 \begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix}$ ✓

It seems that P^n will have a factor 2^{n-1}. If we look at Part 4, this gives a hint for the structure of the remaining matrices.

i.e. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+1 & 2-1 \\ 2-1 & 2+1 \end{pmatrix}$

just enough data

$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2^2+1 & 2^2-1 \\ 2^2-1 & 2^2+1 \end{pmatrix}$

$\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 2^3+1 & 2^3-1 \\ 2^3-1 & 2^3+1 \end{pmatrix}$

– 2 –

2. cont'd

$$\begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix} = \begin{pmatrix} 2^4+1 & 2^4-1 \\ 2^4-1 & 2^4+1 \end{pmatrix}$$

Therefore it appears that the pattern is;

$$P^n = 2^{n-1}\begin{pmatrix} 2^n+1 & 2^n-1 \\ 2^n-1 & 2^n+1 \end{pmatrix}$$ ✓

Check!

$P^{10} =$ (by GDC) $\begin{pmatrix} 524800 & 523776 \\ 523776 & 524800 \end{pmatrix}$

$$= 512\begin{pmatrix} 1025 & 1023 \\ 1023 & 1025 \end{pmatrix}$$

$$= 2^9\begin{pmatrix} 2^{10}+1 & 2^{10}-1 \\ 2^{10}-1 & 2^{10}+1 \end{pmatrix}$$

For $S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

(by GDC)

$$S^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

not enough data generated

$$S^3 = \begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix} = 4\begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix}$$

This looks different, as we could take out another factor of 2.

— 3 —

2. Cont'd

However, $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3+1 & 3-1 \\ 3-1 & 3+1 \end{pmatrix}$

So in $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$ $k = 3$

Now $S^2 = 2 \cdot \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3^2+1 & 3^2-1 \\ 3^2-1 & 3^2+1 \end{pmatrix}$

$S^3 = 4 \begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix} = 2^2 \begin{pmatrix} 3^3+1 & 3^3-1 \\ 3^3-1 & 3^3+1 \end{pmatrix}$

So it appears that the pattern might be;

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$ ✓

Check!

$S^5 = \begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$ (by GDC)

$= 16 \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix}$

$= 2^4 \begin{pmatrix} 3^5+1 & 3^5-1 \\ 3^5-1 & 3^5+1 \end{pmatrix}$

So, in general

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$ (correct)

– 4 –

3. For $M = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

we expect

$$M^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

4. Using the GDC I will try some other values of k and n.

Only one further example checked
No discussion of scope or limitations

$k=5$, $n=7$

$$M^7 = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7$$

by the pattern, this should be;

$$M^7 = 2^6\begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

$2^6 = 64$ $\quad 5^7+1 = 78125+1 = 78126$

$5^7-1 = 78125-1 = 78124$

by GDC;

$$\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7 = \begin{pmatrix} 5000064 & 4999936 \\ 4999936 & 5000064 \end{pmatrix}$$

$$= 64\begin{pmatrix} 78126 & 78124 \\ 78124 & 78126 \end{pmatrix}$$

$$= 2^6\begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

- 5 -

5. Consider $M = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

$$M^2 = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)^2 + (k-1)^2 & (k+1)(k-1) + (k-1)(k+1) \\ (k-1)(k+1) + (k+1)(k-1) & (k-1)^2 + (k+1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} k^2+2k+1+k^2-2k+1 & k^2-1 + k^2-1 \\ k^2-1 + k^2-1 & k^2+2k+1 + k^2-2k+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k^2+2 & 2k^2-2 \\ 2k^2-2 & 2k^2+2 \end{pmatrix}$$

$$= 2\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}$$, which fits the pattern.

Now

$$M^3 = M^2 \cdot M = 2\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 2k^3+2 & 2k^3-2 \\ 2k^3-2 & 2k^3+2 \end{pmatrix}$$

$$= 2^2\begin{pmatrix} k^3+1 & k^3-1 \\ k^3-1 & k^3+1 \end{pmatrix}$$

So, the pattern works !!

This page provides a suitable informal justification

\- 6 -

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student A		
Title of task: Matrix powers		**Type:** (I) II
Date set: 12/03/05	**Date submitted:** 20/03/05	
A Use of notation and terminology	2 / 2	
B Communication	2 / 3	
No introduction, more explanation needed. Hard to follow in places.		
C Mathematical process	4 / 5	
Successful analysis. Only just enough data to warrant level above 2. Only one further example looked at, not enough for 5.		
D Results	3 / 5	
A lack of any discussion of scope or limitations prevents the award of a higher level, even though an informal justification is given.		
E Use of technology	2 / 3	
Not fully exploited. The GDC allows for fuller explanation of other cases.		
F Quality of work	1 / 2	
Satisfactory		

Matrix powers—student B

SL Type I

1. This assignment is looking for a general formula that will give the nth power of the matrix $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

good, (B) concise introduction!

The simplest matrix of this form is

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This table gives results for various values of n

n	M^n	
1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$= \begin{pmatrix} 2^2 & 0 \\ 0 & 2^2 \end{pmatrix}$
3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	$= \begin{pmatrix} 2^3 & 0 \\ 0 & 2^3 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$	$= \begin{pmatrix} 2^4 & 0 \\ 0 & 2^4 \end{pmatrix}$
5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$	$= \begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix}$
10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$	$= \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$
20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$	$= \begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}$

how? GDC? Say so! (E)

The elements of the leading diagonal are powes of 2. The other two elements are zero in every case

$$M^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$$

✓

2. The table gives results for $P^n = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^n$ and $S^n = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^n$

n	P^n	S^n
1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
2	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$ ✓	$\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$ ✓
3	$\begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$	$\begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix}$
4	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$	$\begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$
5	$\begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$	$\begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$

how? GDC? (E) again, say so!

It is more difficult to spot any patterns in these results. To help, the numbers obtained are broken down in the tables below

For P

n	1st element	2nd element
1	3	1
2	$10 = 3\times3 + 1$	$6 = 3 + 1\times3$
3	$36 = 10\times3 + 6$	$28 = 10 + 6\times3$
4	$136 = 36\times3 + 28$	$120 = 36 + 28\times3$
5	$528 = 136\times3 + 120$	$496 = 136 + 120\times3$

Let $P^n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$

then these patterns can be written as

$a_1 = a_1$

$a_2 = 3a_1 + b_1$

$a_3 = 3a_2 + b_2$

$a_4 = 3a_3 + b_3$

$b_1 = b_1$

$b_2 = a_1 + 3b_1$

$b_3 = a_2 + 3b_2$

$b_4 = a_3 + 3b_3$

which leads to the general formula

$$P^n = \begin{pmatrix} 3a_{n-1} + b_{n-1} & a_{n-1} + 3b_{n-1} \\ a_{n-1} + 3b_{n-1} & 3a_{n-1} + b_{n-1} \end{pmatrix}$$

For S

n	1st element	2nd element
1	4	2
2	20 = 4×4 + 2×2	16 = 4×2 + 2×4
3	112 = 20×4 + 16×2	104 = 20×2 + 16×4
4	656 = 112×4 + 104×2	640 = 112×2 + 104×4
5	3904 = 656×4 + 640×2	3872 = 656×2 + 640×4

This generalises to

$$S^n = \begin{pmatrix} 4a_{n-1} + 2b_{n-1} & 2a_{n-1} + 4b_{n-1} \\ 2a_{n-1} + 4b_{n-1} & 4a_{n-1} + 2b_{n-1} \end{pmatrix}$$

3. If $k = 4$, $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = R$

$$R^n = R^{n-1}R$$

$$= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5a_{n-1} + 3b_{n-1} & 3a_{n-1} + 5b_{n-1} \\ 3a_{n-1} + 5b_{n-1} & 5a_{n-1} + 3b_{n-1} \end{pmatrix}$$

✓ good analysis! (C)

If $k = 5$ the matrix is $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} = Q$

$$Q^n = Q^{n-1}Q$$
$$= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 6a_{n-1} + 4b_{n-1} & 4a_{n-1} + 6b_{n-1} \\ 4a_{n-1} + 6b_{n-1} & 6a_{n-1} + 4b_{n-1} \end{pmatrix}$$

This suggests the general result

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_n \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_n \end{pmatrix}$$

4. Let $k = 2$, $n = -1$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3/8 & -\frac{1}{8} \\ -\frac{1}{8} & 3/8 \end{pmatrix}$$

how did you obtain $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$? (B)

Let $k = 2$, $n = -2$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-2} = \begin{pmatrix} 5/32 & -\frac{3}{32} \\ -3/32 & 5/32 \end{pmatrix}$$

$$3 \cdot \frac{5}{32} + -\frac{3}{32} = \frac{12}{32} = 3/8$$

$$\frac{5}{32} + 3\left(-\frac{3}{32}\right) = -\frac{4}{32} = -1/8$$

So the result also holds for negative values of n.

$n = 0$. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$3 \times 1 + 0 = 3 \qquad 1 + 3 \times 0 = 1$

So the result holds for $n = 1$

$3 \times \frac{3}{8} + -\frac{1}{8} = \frac{8}{8} = 1$

$\frac{3}{8} + 3\left(-\frac{1}{8}\right) = 0 = 0$

So result also holds for $n = 0$.

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n$ is undefined if n is not an integer.

The result holds for $n \in \mathbb{Z}$

Suppose k is negative
Let $k = -1$ giving the matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

and let $k = -2$ giving the matrix $\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$

The table shows the first few results

n	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^n$	$\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}^n$
1	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$	$\begin{matrix} 10 & 6 \\ 6 & 10 \end{matrix}$
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$
3	$\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$	$\begin{pmatrix} -28 & -36 \\ -36 & -28 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$
5	$\begin{pmatrix} -32 & -32 \\ -32 & 0 \end{pmatrix}$	$\begin{pmatrix} -496 & -528 \\ -528 & -496 \end{pmatrix}$

These results fit the general formula so it seems that the result holds for negative values of k.

Let $k = \frac{1}{2}$

n	$\begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}^n$
1	$\begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$
2	$\begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix}$
3	$\begin{pmatrix} 9/2 & -7/2 \\ -7/2 & 9/2 \end{pmatrix}$
4	$\begin{pmatrix} 17/2 & -15/2 \\ -15/2 & 17/2 \end{pmatrix}$

good discussion of scope and limitations! (D)

Substituting into the general formula

$\frac{3}{2}\cdot\frac{3}{2} + -\frac{1}{2}\cdot-\frac{1}{2} = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$

$-\frac{1}{2}\cdot\frac{3}{2} + \frac{3}{2}\cdot-\frac{1}{2} = -\frac{6}{4} = \frac{3}{2}$

suggests that the result will also hold for rational values of k.

pursue this! (D)

Let $k = \sqrt{2}$

n	$\begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}^n$
1	$\begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}$
2	$\begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}\begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix} = 2\sqrt{3}+3$
3	

KEIR works!

??

The result holds for $n \in \mathbb{Z}$ and k rational.

5. This is easily justified by matrix multiplication.

$$\begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_{n-1} \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}^n .$$

Satisfactory informal justification (D)

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student B		
Title of task: Matrix powers		**Type:** (I) II
Date set:	**Date submitted:**	
A Use of notation and terminology	2 / 2	
Good use of appropriate notation throughout.		
B Communication	3 / 3	
Well presented. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ result appears without working or explanation. Overall communication, however, is clear and coherent.		
C Mathematical process	5 / 5	
Good analysis. Validity tested.		
D Results	5 / 5	
While the recursive statement does not give an explicit result, it is an acceptable general statement. Discussion of scope and limitations is incomplete, but correct, satisfactory informal justification.		
E Use of technology	2 / 3	
No evidence of use is provided. Limited use only. You could have explored irrational values of *k* with decimal approximations.		
F Quality of work	1 / 2	
Good work. An explicit generalized statement and consideration of irrational values of *k* would have demonstrated greater insight.		

Stopping distances—student C

SL Type II

Stopping Distances

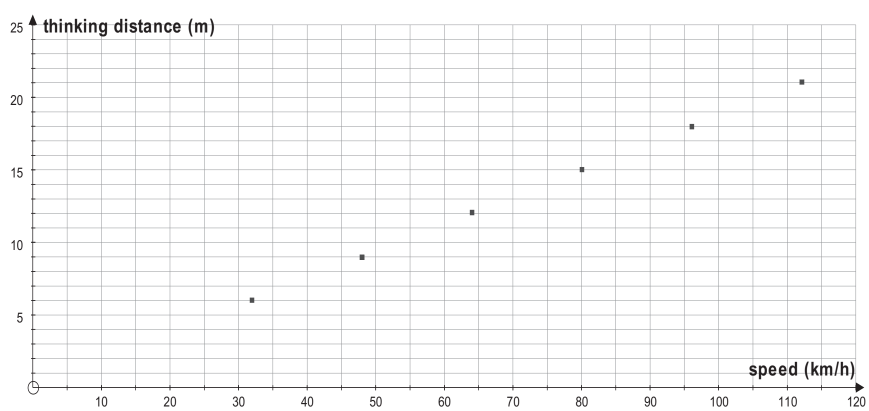
In this assignment I will be building up a model for the relationship between speed and overall stopping distance for cars.

The table below gives average times for Thinking Distance and Braking Distance at various speeds.

Speed (kmh^{-1})	Thinking distance (m)	Braking distance (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

Source:

Let s kmh^{-1} be the speed and d metres be the thinking distance. The graph below is of d against s.



This looks like a straight line. Considering the first two points, its gradient is

$\frac{9-6}{48-32}=\frac{3}{16}$. Also when the speed is zero the thinking distance will also be zero.

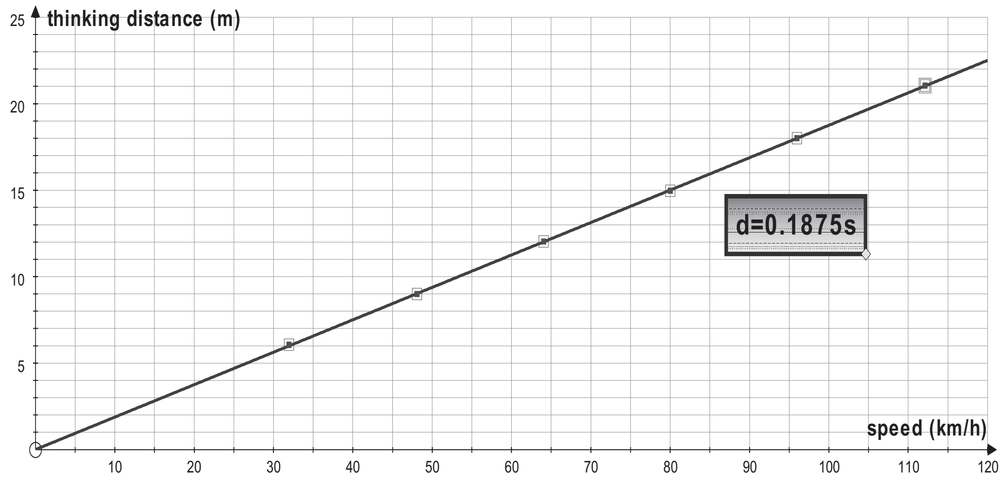
The equation of the line can now be found

$$d-0=\frac{3}{16}(s-0)$$

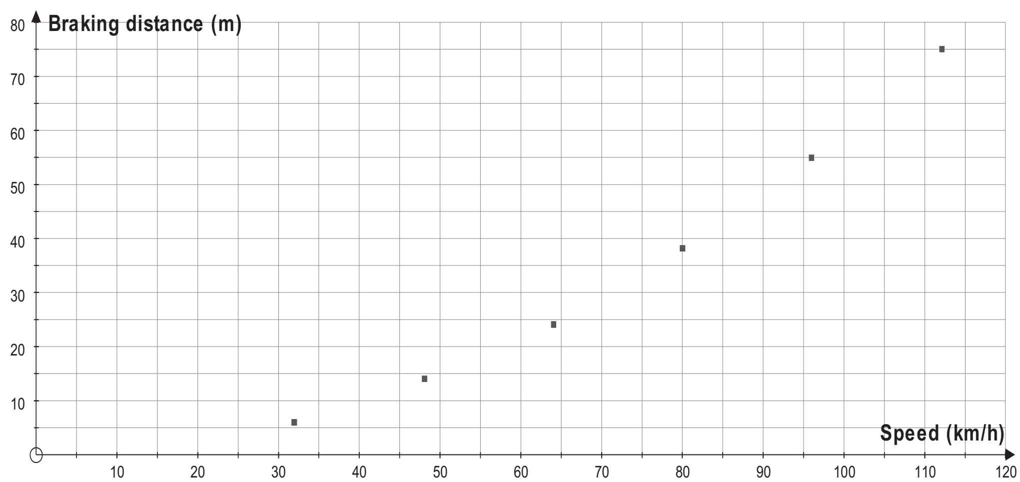
$$d=\frac{3}{16}s$$

1

The graph below shows the graph of this function and the original data. The fit is excellent.



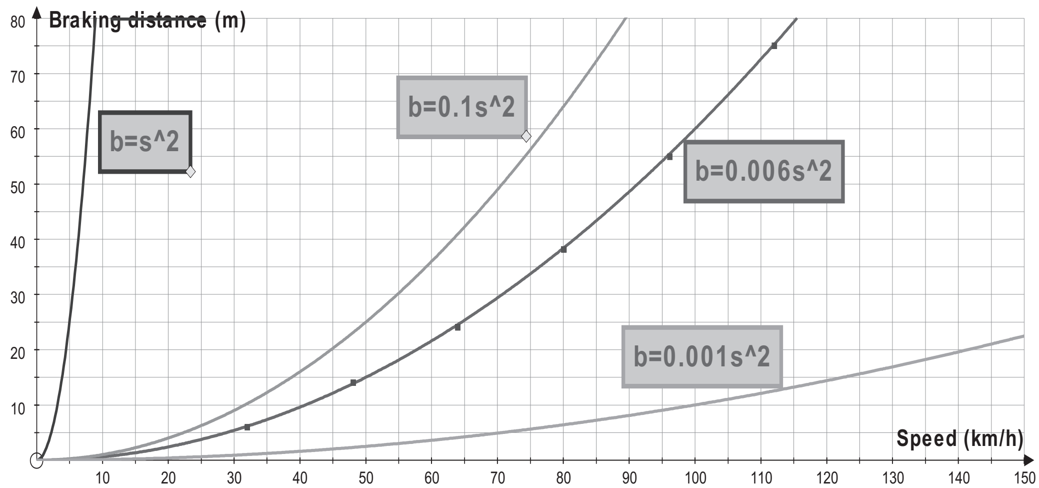
Now I will consider the braking distance data.
Let b metres be the braking distance. The graph below is of b against s.



This looks like the general shape of part of a quadratic function with its vertex at (0,0) The means that the function has the form $b = as^2$. Using Autograph, I tried to find a suitable value of *a*. As *a* changes the graph is stretched vertically with scale factor *a*. I first drew $b = s^2$ to see whether *a* was greater than or less than one. Clearly it is considerable less than one. Using trial and error I found that

2

0.006 was a suitable value of *a*. The diagram below shows a few of the values I tried.



So a suitable function to model braking distance is
$b = 0.006s$

Now I will try to find a model for the overall stopping distance.

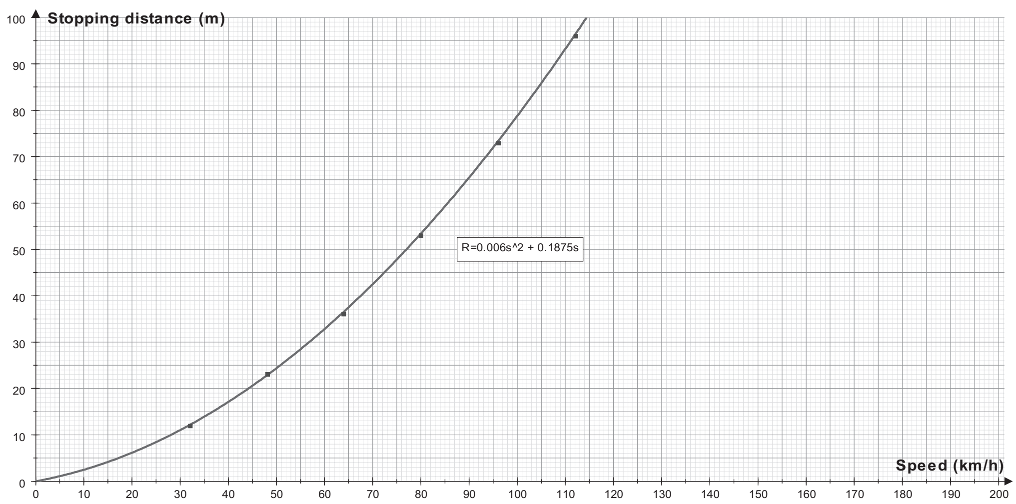
The table below gives the relevant values

Speed (kmh^{-1})	Stopping distance (m)
32	12
48	23
64	36
80	53
96	73
112	96

Since we add the thinking distance to the braking distance to get overall stopping distance we should add the expressions for these to get a formula for the stopping distance. The graph shows the data from the table above and the graph of the function.

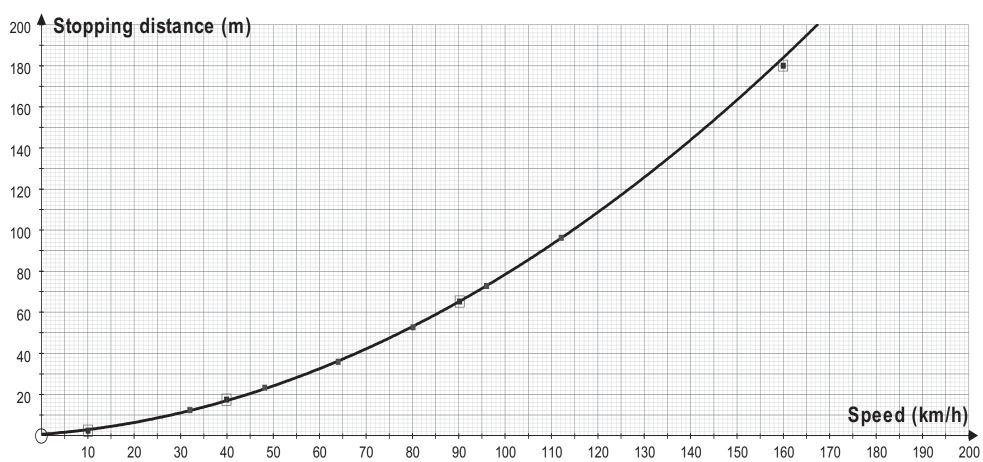
$$R = 0.006s^2 + \frac{3}{16}s\,.$$

3



The model fits the data well for the range of speeds given.

Additional data, including speeds outside this range is given.
The next graph includes this data.



It can be seen that the model still appears to be good. The value for a speed of 160kmh^{-1} is slightly off the curve and so the fit may not be so good for higher speeds. More data would have been helpful to test this further.

4

As Autograph can fit curves to data I thought it would be interesting to see what function it came up with. The function it produced was

$$R = 0.006138s^2 + 0.1643s + 0.6.$$

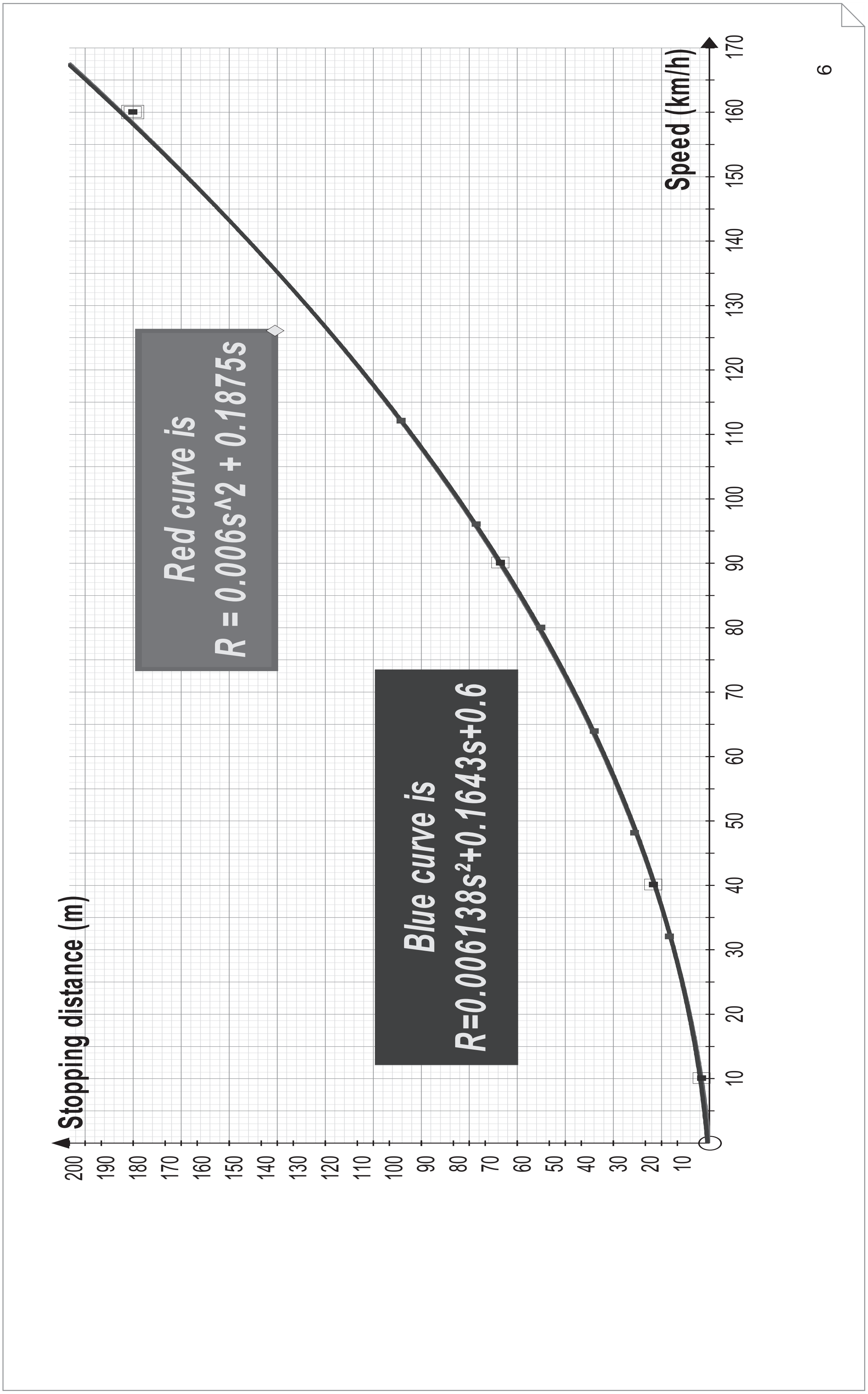
This is compared with the function I have found on the following graph. Even when enlarged the curves are almost identical, so both functions fit the data well. However, I prefer my model as it is built up of the two parts of the overall stopping distance and also it gives R=0 when s=0 which it what really happens.

The original data was for cars. It is likely that heavier vehicles would require longer stopping distances as their momentum would be greater for the same speed. So the model should only be used for cars. Even for cars other factors might affect the stopping distance. It is well known that when the roads are wet there is less friction between the tyres and the road surface and stopping takes longer distances. Rough surfaces would need less distance. Probably the data was collected from vehicles with brakes in top condition. Older vehicles with less good brakes would not perform so well. Who was driving the cars? Older people generally have slower reaction times so would need longer thinking distances.

The model fits the data well. So it will be useful for working out stopping distances for cars but only on the kind of road surface used when data was collected and with cars and drivers with ages similar to those that took part in the test.

Software used: Autograph (version3) www.autograph-maths.com

5


6
Stopping distance (m)
Speed (km/h)
Red curve is
R = 0.006s^2 + 0.1875s
Blue curve is
R=0.006138s²+0.1643s+0.6

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student C		
Title of task: Stoppping distances		**Type:** I (II)
Date set:	**Date submitted:**	

A Use of notation and terminology 2 / 2

Good overall. One small error where s is used instead of s^2 but this is allowed.

B Communication 3 / 3

The additional data (page 55) and the comparison of the two graphs (the model and the regression) should be placed in context. The general level of communication, however, is excellent.

C Mathematical process 5 / 5

Variables are defined. Parameter “a” is successfully determined. Model function fit is considered and model function is applied to additional data.

D Results 4 / 5

Reasonableness of the model had been carefully considered. Two models have been compared, but limitations have not been considered. The degree of accuracy used throughout is appropriate even though not discussed.

E Use of technology 3 / 3

Excellent use of Autograph® software. Graphs are used effectively to enhance understanding.

F Quality of work 2 / 2

Excellent work.

Modelling the amount of a drug in the bloodstream—student D

SL Type II

In this assignment I will investigate the absorption of a drug into the human bloodstream by creating a model function that matches the supplied data. I will also investigate how regular doses of the drug cause the overall quantity of drug in the bloodstream to change with time.

To creat a model function I will first create a table of values by reading coordinates from the supplied graph. I will use the variable t to represent the time elapsed, in hours. I will use the variable A to represent the amount of drug in the bloodstream, in µg. Clearly t and A will take on only positive or 0 values.

Part A
The following data was read from the supplied graph, and is accurate to the nearest tenth of a unit. Once again, t is in hours, and A is in µg.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10
A	10	9	8.3	7.8	7.2	6.7	6	5.3	5	4.5	4.3	4	3.7	3	2.8	2.5	2.5	2.1	1.9	1.8	1.5

The task states that the rate of decrease is approximately proportional to the amount of drug remaining in the bloodstream. This suggest that each subsequent amount is a fraction of the previous measured amount. This would mean that this is a decay model. Such situations can be modelled by a function of the form $A = A_0 e^{\lambda t}$, where A_0 is the initial quantity of drug administered. I know that A_0 must be 10µg, since this is the value of A at $t = 0$. Thus we have $A = 10e^{\lambda t}$. We must now determine a suitable constant, λ.

If I substitute the coordinates (t, A) from the graph I can obtain individual values of λ for each pair as follows;

$(t, A) = (3.0, 6)$

$$6 = 10e^{3\lambda}$$

$$0.6 = e^{3\lambda}$$

$$\ln(0.6) = 3\lambda$$

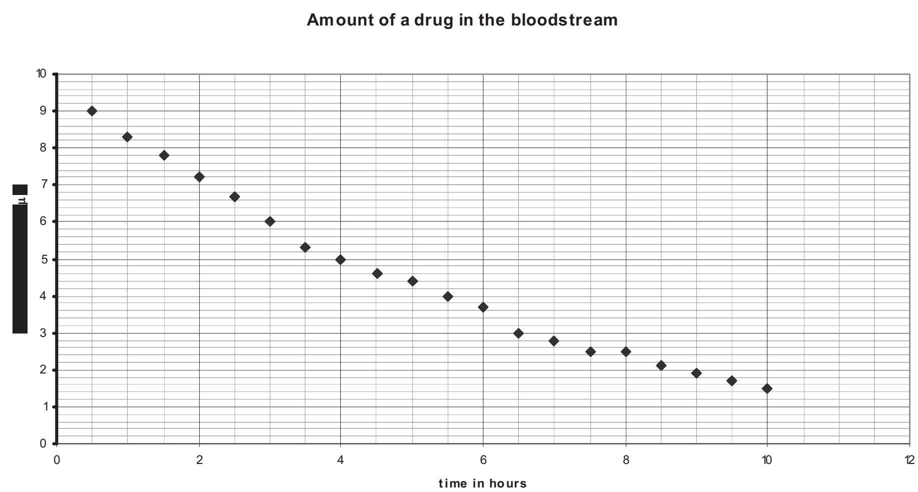
$$\lambda = \frac{\ln(0.6)}{3} \cong -0.170 \quad 3 \text{ s.f.}$$

However, this is only a single instance, and will not be representative of the actual value of λ. A better estimate can be obtained by calculating the value of λ for each pair of coordinates, and then averaging these values. By using the same method as above I calculated λ-values for each point. The calculated values of λ are presented below, rounded to 3 s.f.. Note that a value for λ cannot be found for $t = 0$ since division by 0 is undefined. The average value of λ was found using more accurate calculated values and then rounding at the end.

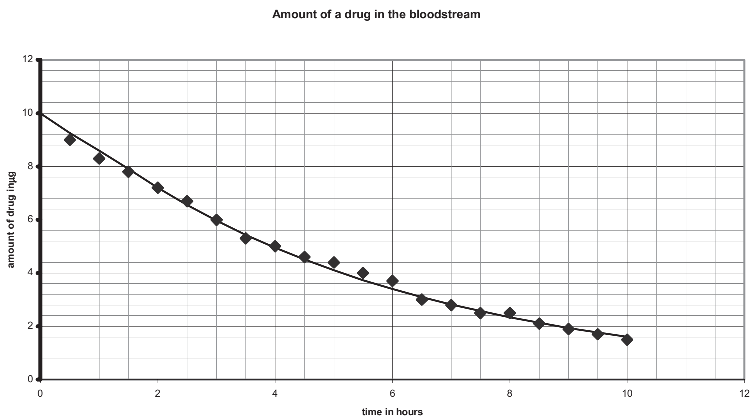
λ = {NA, -0.211, -0.186, -0.166, -0.164, -0.160, -0.170, -0.181, -0.173, -0.177, -0.169, -0.167, -0.166, -0.185, -0.182, -0.185, -0.173, -0.184, -0.185, -0.180, -0.190}

$\overline{\lambda} \cong -0.178$

I can now use the function $A = 10e^{-0.178t}$ to approximately model the data. The graph below shows the set of points (t, A) as supplied. The second graph shows the data and the model function.



Data Plot of Amount *vs* Time

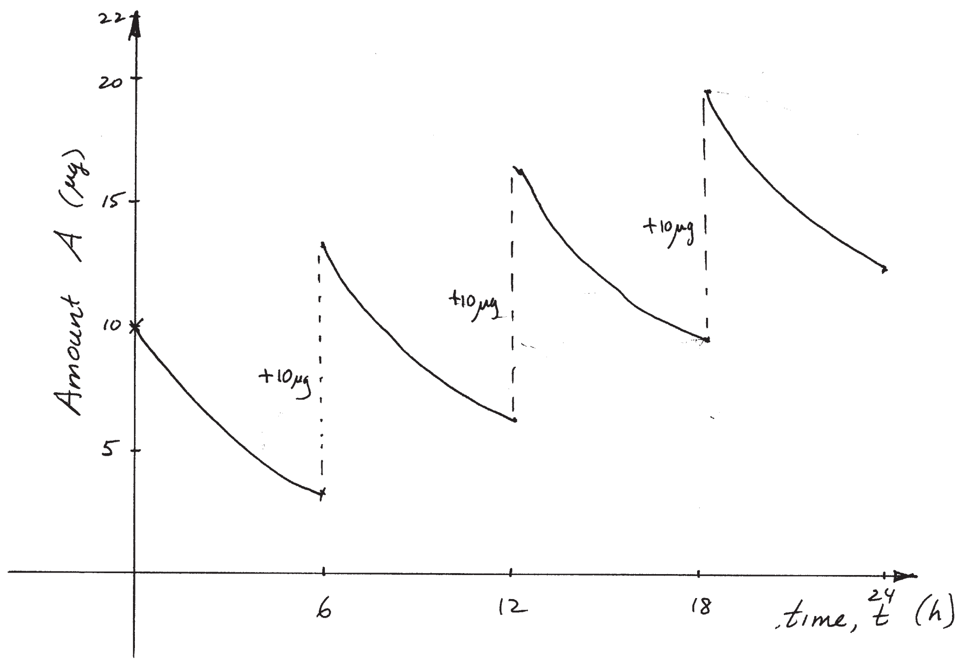


Data Plot and $A = 10e^{-0.178t}$

The function fits the data quite well and some discrepancy is to be expected due to the approximate measures involved. The function follows the same behaviour as the data, although the real-life situation does not include negative values of time. The function continues past $t = 10$ and approximates what will happen to the amount of drug remaining in the bloodstream as time continues to pass. Note that the amount is tending towards 0, although the model function does not allow the amount to ever really reach 0. In reality I would expect that the drug eventually disappears from the bloodstream altogether.

Part B

Now I will investigate how the amount of drug in the bloodstream changes when a 10µg dose is taken every 6 hours. Clearly the amount of drug in the blood will decrease according to the model above, but jump up 10 units every 6 hours. I expect that the graph would look like the sketch below.



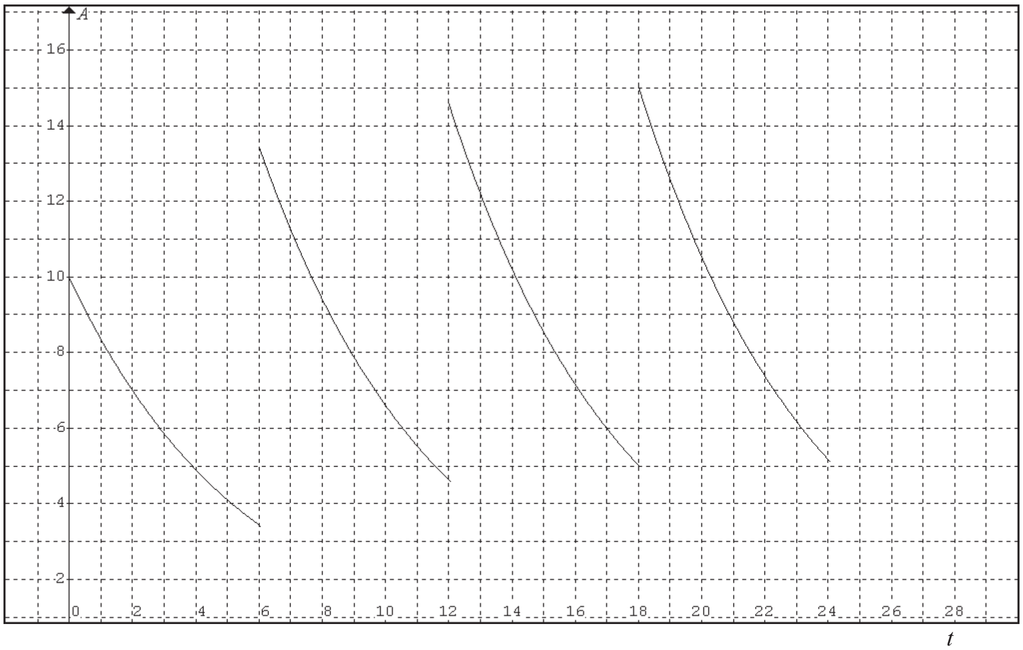
Here I assume that the new dose is immediately and completely absorbed into the bloodstream, or else the increase in the amount would be gradual. I would also expect that the amount remaining after 6 hours from the first dose would add to the total amount in the blood. So, in fact, the amount after 6 hours (and for subsequent additional doses) would include the residual amount from the first dose plus the amount left from the second dose. Thus the pieces of the graph above would be comprised of a sum of versions of the model function, adjusted for a time translation.

That is, for $6 \le t < 12$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)}$

and for $12 \le t < 18$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)} + 10e^{-.178(t-12)}$

and for $18 \le t < 24$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)} + 10e^{-.178(t-12)} + 10e^{-.178(t-18)}$

Using Graphmatica, I graphed these functions. The results are shown below.

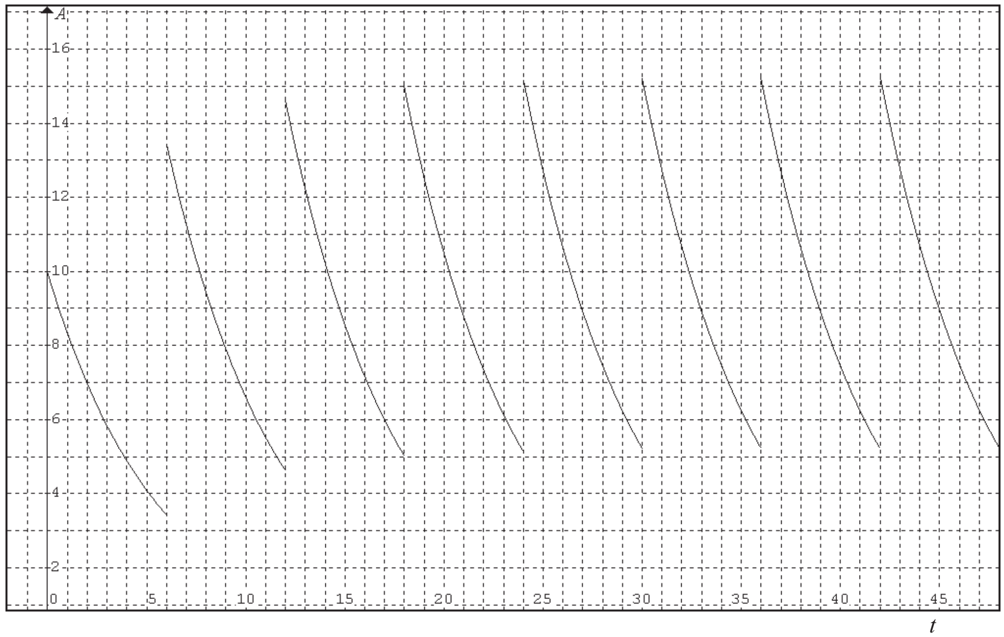


y=10*e^(-.178(x-18))+10*e^(-.178(x-12))+10*e^(-.178(x-6))+10*e^(-.178x) {18,24}

The minimum value is approximately 3.8µg, while the maximum is approximately 15µg.

If no further doses were taken I would expect that the amount of drug in the bloodstream would decrease in a fashion similar to how the original model function decreased, as there is no new supply. Eventually there would be no drug left in the blood system.

If doses were continued to be taken every 6 hours, it appears that the maximum amount would stabilize, as would the minimum. The cummulative effect of the previous doses would decrease as time goes by due to the exponential rate of decay. This would mean that the minimum (and thus the maximum since this is just the previous minimum increased by 10) would tend to a stable value. I have graphed the longer term behaviour below, and this confirms my conjecture.



This behaviour makes sense medically, as a doctor would want the level of a medication in a patient to stabilize within certain safe and effective values, and not fluctuate wildly over time.

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student D		
Title of task: Modelling the amount of a drug in the bloodstream		**Type:** I (II)
Date set:	**Date submitted:**	

A Use of notation and terminology 2 / 2

Correct throughout including appropriate use of “approximately equals” sign

B Communication 3 / 3

Excellent communication. Explanations are clear and well-supported by good graphs. The piece can be read without reference to the statement of the task.

C Mathematical process 5 / 5

Variables are clearly defined. The parameters of the function are linked to the context. A suitable model is formulated and student considers how well it fits the data. The initial model is adapted for repeated doses.

D Results 4 / 5

The results are correctly interpreted in context and the degree of accuracy used is appropriate. Some limitations of the model are discussed on page 61. Some major simplifications are mentioned (immediate absorption) but not referred to when considering the overall quality of the model. The final part is imprecise and not completely correct.

E Use of technology 3 / 3

Developing the piecewise function is critical to the development of the task.

F Quality of work 1 / 2

Not quite enough for a level 2. A more precise mathematical analysis of final graph would have made this a level 2, for example, an attempt to fit a function to a graph of the maximum values.